circumferential stress indicates that the flexural stress contribution amounts to a maximum value of 15% of the membrane stress in that direction and is therefore not plotted. Klosner and Levine,[7] in comparing the elasticity and shell theory solutions, have found that for very short shells subjected to discontinuous transverse normal loads, even higher-order shell theories such as the one used here can give inaccurate results for the stresses. For this reason, the minimum value of L/a has been restricted to 0.5 in Fig. 5. Moreover, since the loads considered here are continuous, the stresses computed by Reissner-Naghdi theory should be fairly accurate.

Conclusions

Using Reissner-Naghdi shell theory, it is found that substantial reduction in flexural stresses can be effected by supporting the edges elastically. Moreover, for support modulus values as large as 10^5 psi the edge moment is almost zero and the maximum value of bending moment occurs between the edge and midlength of the shell. It is also found that the support width should be kept as small as practical in order to keep the magnitude of flexural stresses low.

Finally, a comparison of classical vs shear deformation theory indicates that although for thin shells the two theories predict almost identical moment and shear resultant, the classical theory overestimates stresses by about 35% for short, thick shells.

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Kinetic Factors in Heterogeneous Combustion of Cross-Linked Polymers

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Nomenclature

A = Arrhenius frequency factor, sec^{-1}
B' = B, thermochemical mass transfer number
C_p = specific heat, cal/g°C
D = instantaneous duct diameter, cm
E = energy of activation in thermal degradation, Kcal/mole
G_o = initial oxidizer rate, g/cm^2 sec
G = total mass flow rate, g/cm^2 sec
K = thermal conductivity, cal/cm sec°C
L = length of the grain, cm
$\dot{m}$ = mass regression rate, g/cm^2 sec
$\dot{r}$ = linear regression rate, cm/sec
Re_x = Reynolds number based on x
T = temperature, °K
ρ = density, g/cm^3
α = thermal diffusivity, cm^2/sec
$\left(\frac{\bar{\rho}}{\rho_e}\right)$ = semiempirical variable property correlation factor

Subscripts
o = initial
s = surface
f = fuel

Introduction

THE heat and mass transfer within the turbulent boundary layers has been investigated by Marxman et al.[1–3] in heterogeneous combustion of linear polymers. The regression rate of similar polymers has been correlated by Houser[4] and Rabinovitch[5] with kinetics of degradation in solid phase. In all these investigations the polymers are derived from vinyl monomers, where considerable mechanical strength is reduced with time. In this investigation more precise data have been obtained in heterogeneous combustion of cross-linked polyesters to study the effect of kinetic factors in solid phase.

Experimental

A. Preparation of Polymers

Four polyesters were prepared by reacting propylene glycol with a suitable ratio of maleic anhydride and phthalic anhydride. The composition of linear polyester is varied by varying the ratio of maleic and phthalic anhydride. The linear polyester resin is crosslinked with styrene. The molar ratio of maleic anhydride to styrene is selected either 1:1 or 1:2. The molar ratio of maleic anhydride, phthalic anhydride, and styrene in polymer is shown as M:P:S, respectively, and the physical properties are tabulated in Table 1.

B. Degradation Studies

The experimental procedure of Rastogi and Gupta[6] was adopted for finding the degradation constant. The least-square technique was used to evaluate the degradation rate constant, Arrhenius factor, and energy of activation. The experimental data were processed in IBM-360-44 computer and the mean values of kinetic factors are given in Table 1.

C. Regression Rate and Surface Temperature

The cylindrical grains (o.d. 5 cm, length 8.5 cm) having an L/D_o ratio around 10 were used for combustion in a tubular burner of similar dimensions. The diameter of circular inlet for gases is around 0.7 cm and flow of gases was controlled by a three-way valve in between flowmeter and burner. The heterogeneous reaction was started by igniting the fuse wire electrically by Parr Ignition Unit. The commercial oxygen was used for combustion and nitrogen for extinguishing the flame. The exit end of burner was open to atmosphere. The thermocouple, prepared from copper-constantan wires,‡ was inserted in the center of the grain and its leads were connected to a strip-chart recorder.§ The typical temperature-time traces until the thermocouple is in solid phase are shown in Fig. 1. The average linear regression rate was calculated by measuring the mean radius before and after the combustion for a period of time. The mass regression rate was calculated by finding loss in weight during combustion. The time of combustion was usually selected between 20–60 sec to assure the steady-state conditions are attained at the end of the run. If heat of pyrolysis of polymer is

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Index categories: Combustion in Heterogeneous Media; Fuels and Propellants, Properties of.

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‡ 0.001 in. Omega Engineering Inc., Stamford, Conn.

§ Honeywell: 0–10 mv.

Table 1 Physical and chemical properties of polymers

Polymer No.	Polymer Compn. M:P:S	E kcal/mole	$A \times 10^{-3}$ Sec^{-1}	$\alpha \times 10^4$ cm^2/sec	ρ g/cm^3	$K \times 10^4$ cal/cm sec °C	C_p Cal/g
1	0.4:0.6:0.8	15.3	1.0	12.743	1.20	5.20	0.34
2	0.5:0.5:0.5	16.3	1.5	12.285	1.20	5.16	0.35
3	0.5:0.5:1.0	15.3	1.8	12.285	1.20	5.16	0.35
4	0.6:0.4;0.6	16.6	1.1	13.022	1.21	5.24	0.33

Table 2 Surface temperature and regression rate of polymers

No.	$\dot{r} \cdot 10^3$ cm/sec	T_s°K	G_o g/cm^2 sec	G g/cm^2 sec	$\dot{m} \times 10^3$ Experimental g/cm^2 sec	$\dot{m} \times 10^3$ Eq. (2) g/cm^2 sec	$\dot{m} \times 10^3$ Eq. (3) g/cm^2 sec
1	7.5	822±5	1.41	1.74	13.5	2.6308	12.0
2	9.3	862±5	1.41	1.78	16.8	5.6757	12.0
3	4.8	868±5	0.43	0.54	7.0	11.9430	4.0
4	4.0	812±5	0.28	0.38	5.5	2.0604	3.0

neglected, then surface temperature can be calculated from the following equation[4]:

$$T - T_o = (T_s - T_o)\exp(-\dot{r}x/\alpha) \qquad (1)$$

The average values of surface temperature are given in Table 2.

Discussion

The errors shown in the surface temperature are most probably due to errors in measuring the exact distance from the burning surface. Care was taken to obtain the repetitive results in several runs but it was observed that the thermocouple mounting in the grain plays an important role in the nature of the time-temperature trace. Similar observations were also reported by Krishnamurthy and Williams.[7] The average values of T_s are used for the calculation of mass regression rate from the following equation[4]:

$$\dot{m} = A\rho_f \Sigma \exp(-E/RT_s)dx \qquad (2)$$

Fifty slices of 1×10^{-3} cm were used in calculation and ambient temperature of each slice was obtained from Eq. (1). The calculations were carried out on IBM-360-44 computer. The mass regression rate is also calculated by the following equation[3]:

$$\dot{m} = 0.036\left(\frac{\bar{\rho}}{\rho_e}\right) G B'^{0.23} Re_x^{-0.2} \qquad (3)$$

The values of $(\bar{\rho}/\rho_e)$ and B' are assumed unity and 10, respectively, in the calculation. Since the value of B' is around 10 for boric acid polyester[8] (a rough calculation indicates that the value for this polymer is around 11 and more precise value is being determined), hence it would not be unfair if this value is used, because the polymers have similar structure. The calculated values of mass regression rate from Eqs. (2) and (3) are tabulated in Table 2.

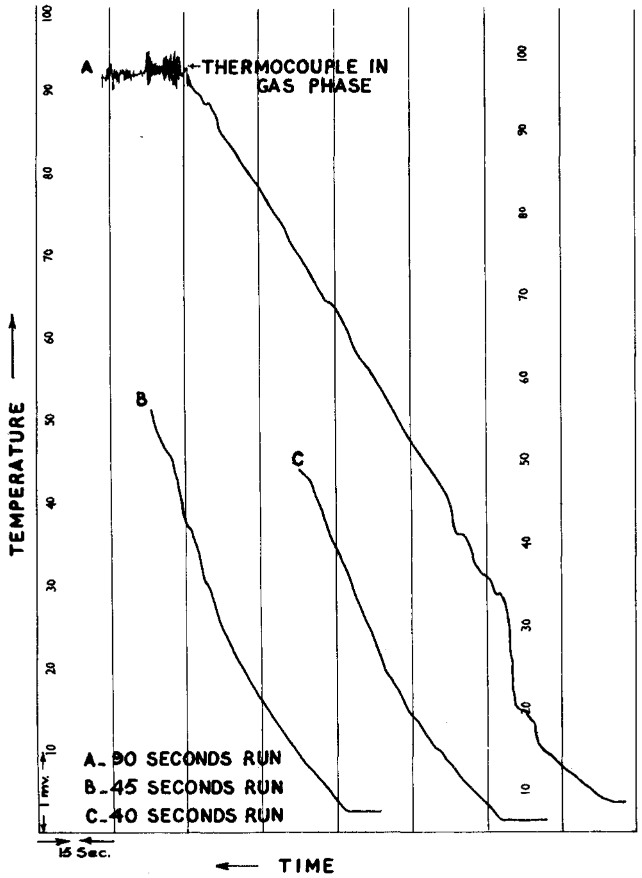

Fig. 1 Typical time-temperature traces.

It is apparent from the Tables 1 and 2 that there is no definite trend of surface temperature with molecular changes and, in combustion, the heat and mass transfer mechanism within the boundary layers is a predominant factor. During the degradation studies it was noticed that at higher temperatures (< 600°C) there is a formation of liquid like film around solid mass, which indicates that first crosslinks are broken and then scission of linear chains into smaller fragments. By examination of time-temperature traces obtained, there is very little possibility of liquid-like film as in PMMA[7] and in other vinyl polymers. There is a charring at the surface which indicates that the polymer is breaking into larger molecular groups than predicted by thermal degradation data. This may be the reason that kinetic factors of solid phase are not playing the predominant role. More extensive data are needed to explore the changes of surface temperature with oxidizer flow rate and experiments are being carried out.

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Necessary Condition for Piecewise Uniform Optimum Design under Frequency Constraint

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Introduction

WE have shown in a recent Note,[1] how the piecewise uniform minimum weight of a clamped bar could be obtained exactly for any chosen fundamental frequency of axial vibration. It was also shown that the continuous optimum design can be approximated as closely as desired by using an appropriate number of uniform regions. Since it is desirable from a practical point of view to use as few steps as possible, the objective of this Note is to show the minimum number of steps one must use in order to obtain a given fundamental frequency.

We use here the same notation as in Ref. 1, and the reader should refer to this Note for the derivation of the equations. However, we assume now that the n uniform regions have the same length $L_i = L/n$ (Fig. 1), the frequency parameters α_i all being equal to $\alpha = \beta L/n$. Besides, the only nonstructural mass M is located at the bar tip.

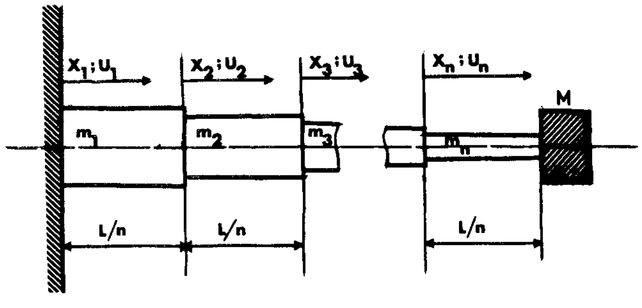

Fig. 1 Clamped bar with *n* uniform elements.

Relation between Frequency and Number of Steps

To obtain information about α, we must eliminate the Lagrange multipliers from Eqs. (7–10) of Ref. 1. Elimination of λ_{i-1} from the last two leads to relation

$$\lambda_i = l \sin^2(\alpha+\Phi_i)/\sin\alpha\cos(\alpha+2\Phi_i) \quad (i = 2,\ldots,n) \qquad (1)$$

which, combined with λ_{i-1}, from Eq. (10), Ref. 1, requires that Φ_i and Φ_{i-1} satisfy

$$\cot\Phi_i = [\sin\alpha\sin(\alpha+\Phi_{i-1}) \pm \cos\Phi_{i-1}]/\cos\alpha\sin(\alpha+\Phi_{i-1}) \qquad (i = 3,\ldots,n) \qquad (2)$$

Since $0 < \Phi_i < \pi/2$, the plus sign must be chosen in Eq. (2). This, together with special consideration of the case $i = 2$, (with $\Phi_i = 0$) leads to the alternate recurrence relations

$$\cot\Phi_i = \tan\alpha + 1/(\sin\alpha\cos\alpha + \cos^2\alpha\tan\Phi_{i-1}) \qquad (i = 2,\ldots,n) \qquad (3)$$

If $0 < \alpha < \pi/2$, we see from Eq. (3) that

$$\cot\Phi_i > \tan\alpha \quad (i = 2,\ldots,n)$$

Therefore, $\Phi_i + \alpha < \pi/2$, which is the condition to have the first mode. Thus, the only condition in order to get the required fundamental frequency with a given number of steps n is that

$$\alpha = \beta L/n < \pi/2 \qquad (4)$$

Equation (4) shows that for a given number of steps n, the maximum frequency factor obtainable is

$$\beta L \leqq n\pi/2$$

Conversely, if we require a frequency factor βL, we need a minimum number of steps: $n > 2\beta L/\pi$. In particular, as we might expect, a uniform member, $n = 1$, has a maximum $\beta L = \pi/2$.

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Index categories: Structural Dynamic Analysis; Optimal Structural Design.

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Exact and Approximate Analyses of Transient Wave Propagation in an Anisotropic Plate

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Introduction

MOST of the simplified theories for plates are constructed based on the assumption that the characteristic length of deformation is much larger than the thickness of the structure. The conventional approach in determining the region of adequacy of the plate theories has been by the use of comparing the dispersion curves for the harmonic wave propagation in the plate obtained from the plate theories and the three-

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